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SOME CLASSES OF TWO-DIMENSIONAL STATIONARY VORTEX STRUCTURES IN AN IDEAL LIQUID

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New types of plane stationary vortex formations in an ideal liquid are found. These structures are described by exact solutions of the equation for the stream function. This equation is the elliptical analog of the well-known Bullough–Dodd–Gibert–Shabat nonlinear wave equation. The Lyapunov stability of some of the solutions follows from Arnol'd's theorem.

Vortex structures, which are of undoubted independent interest, attract additional interest in connection with the study of large-scale formations in turbulent motions. A number of representations for such structures in a liquid and a plasma have been found in the last decade [1–4]. In the case of plane stationary flows of an ideal incompressible liquid, exact solutions of the equations have been found for the stream function

$$\Delta\psi = \omega(\psi). \tag{1}$$

The most progress has been achieved in the investigation of Eqs. (1) with right sides $\sin\psi$ and $\sinh\psi$. This is because the hyperbolic analogs of these equations have been studied fairly well by Hirota's method, the method of inverse scattering problems, and finite-zone integration. Among nonlinear wave models, the Bullough–Dodd–Gibert–Shabat method occupies a special place. It should be remembered, however, that at the start of the century this equation was analyzed in work on differential geometry [5]. Despite this fact, no equation for an N-soliton solution is found in the literature.

In the present paper, the vortex structures correspond to solutions of the equation

$$\Delta\psi = \delta\exp(\psi) - \exp(-2\psi), \qquad \delta = \pm 1. \tag{2}$$

To construct solutions of Eq. (2), we make the change $\psi = \ln(\delta u)$ and we seek u in the form $u = h/g$. Such a representation of solutions is typical in the theory of solitons [6]. As a rule, the functions h and g are written in the form of a finite number of terms of the type $r_i\exp(a_ix + b_iy)$, where r_i, a_i, and b_i are certain coefficients. The main difficulty lies in finding these coefficients. An attempt to obtain equations for the coefficients was made by Markov [7].

To find the functions h, g and g, we use the system of bilinear equations

$$h^2\Delta(\ln h) = h^2 - g^2; \tag{3}$$

$$g^2\Delta(\ln g) = -hg + g^2, \tag{4}$$

each solution of which generates a solution of Eq. (2) for $hg \neq 0$. This is easy to ascertain by dividing (3) by h^2 and (4) by g^2 and taking the difference of the resulting equations.

Taking $g = \tau^2$, from Eq. (4) we find the representations for h and u:

$$h = \tau^2 - 2(\tau\Delta(\tau) - (\nabla\tau)^2), \qquad u = 1 - 2\Delta(\ln\tau). \tag{5}$$

Substituting this expression for h into (3), we obtain an equation for the function τ. We denote this as the H equation, it being the analog of the well-known bilinear equations for the τ-function in Hirota's formalism [6].

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The difference is that in this case, the H equation is trilinear and very cumbersome. We note that, in contrast to the approach suggested by Markov [7], to construct solutions of Eq. (2) one must find one function τ. The most important, however, is the fact that particular solutions of the H equation have the same structure as N-soliton solutions of Hirota's bilinear equations [6]. It is easy to ascertain by direct substitution, for example, that the function $\tau_1 = 1 + s\exp(kx \pm \sqrt{3-k^2}y)$, where $s, k \in R$, satisfies the H equation.

Being guided by the analogy with equations for two-soliton solutions, we can seek an exact solution of the H equation in the form

$$\tau_2 = 1 + f_1 + f_2 + p_{12} f_1 f_2, \tag{6}$$

where

$$f_i = s_i \exp(k_i x \pm \sqrt{3-k_i^2}y), \qquad s_i, k_i \in R, \tag{7}$$

and the quantity p_{12} is to be determined. Substituting τ_2 into the H equation, we obtain a polynomial in f_1 and f_2. Equating the coefficient of the polynomial for $f_1^3 f_2^3$ to zero, we find

$$p_{12} = \frac{4m_1m_2k_1k_2 - 9m_1m_2 + 4k_1^2k_2^2 - 6k_1^2 - 9k_1k_2 - 6k_2^2 + 27}{4m_1m_2k_1k_2 + 9m_1m_2 + 4k_1^2k_2^2 - 6k_1^2 + 9k_1k_2 - 6k_1^2 + 27}, \tag{8}$$

where $m_i^2 = 3 - k_i^2$. We then verify that τ_2 with the value of p_{12} found does satisfy H.

It is now natural to assume that the standard equations for N-soliton solutions [6] hold in this case. For small N, a direct test is possible using the REDUCE system of analytical calculations, but in general an N-soliton equation must be verified by the method of mathematical induction. For completeness of the presentation, we give the solutions τ_3 and τ_4:

$$\begin{aligned} \tau_3 &= 1 + f_1 + f_2 + p_{12}f_1f_2 + p_{13}f_1f_3 + p_{23}f_2f_3 + p_{12}p_{13}p_{23}f_1f_2f_3, \\ \tau_4 &= 1 + \sum_{1\leqslant i\leqslant 4} f_i + \sum_{1\leqslant i<j\leqslant 4} p_{ij}f_if_j + \sum_{1\leqslant i<j<l\leqslant 4} p_{ij}p_{il}p_{jl}f_if_jf_l + p_{12}p_{13}p_{14}p_{24}p_{34}f_1f_2f_3f_4, \end{aligned} \tag{9}$$

where f_i are given by Eq. (7) and p_{ij} by Eq. (8) with the subscripts 1 and 2 replaced by i and j.

In addition to the solutions τ_i given above, there are others that are also expressed in terms of elementary functions. To construct such solutions, it is convenient to use differential relationships of a special kind. Note that the solution (6) satisfies the linear differential equations with constant coefficients

$$d_x(d_x - k_1)(d_x - k_2)(d_x - k_1 - k_2)\tau = 0; \tag{10}$$

$$d_y(d_y - m_1)(d_y - m_2)(d_y - m_1 - m_2)\tau = 0, \tag{11}$$

where d_x and d_y are derivatives with respect to x and y, respectively, and $m_i^2 = 3 - k_i^2$. It turns out that there are solutions of Eqs. (10) and (11) that are solutions of the H equation and do not coincide with (6).

We give two examples. If we have $k_1 = k_2 = 0$, then the function

$$1 + r_1 x \exp(\sqrt{3}y) + (r_1/6)^2 \exp(2\sqrt{3}y), \qquad r_1 \in R$$

satisfies Eqs. (10) and (11) and H. For $k_1 = k_2 = \sqrt{3}/2$, a solution of these equations is the function

$$1 + r_1 \exp(x\sqrt{3}/2 + 3y/2) + r_2 \exp(\sqrt{3}x), \tag{12}$$

where r_1 and r_2 are arbitrary constants.

Examples of this type are constructed as follows. We first find a general solution of Eqs. (10) and (11) that contains some arbitrary constants. Then substituting this general solution into the H equation, we obtain additional conditions on the constants.

It is simple to write the differential relationships that the functions τ_n satisfy. These relationships for an arbitrary n are

$$d_x \prod_{1\leqslant p\leqslant n} \Big(\prod_{1\leqslant i_1<\ldots<i_p\leqslant n} (d_x - k_{i_1} - k_{i_2} - \ldots - k_{i_p}) \Big)\tau = 0,$$

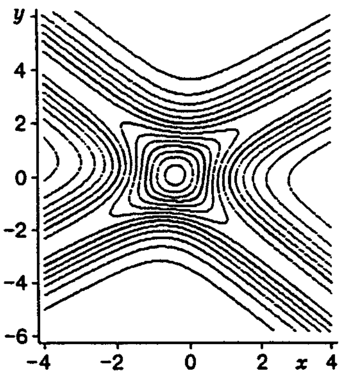

Fig. 1

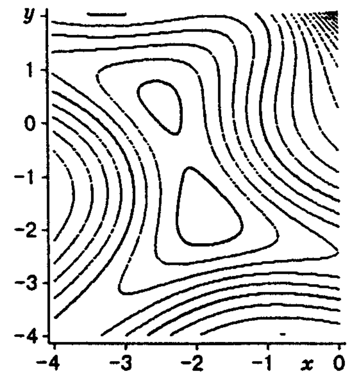

Fig. 2

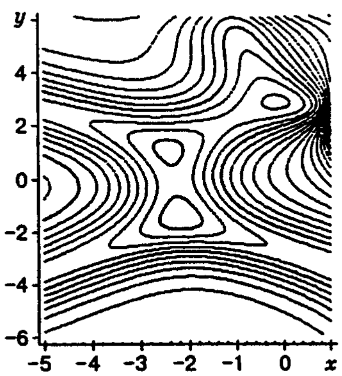

Fig. 3

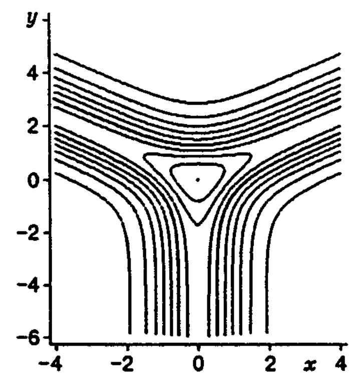

Fig. 4

$$d_y \prod_{1\leqslant p\leqslant n} \Big(\prod_{1\leqslant i_1<\ldots<i_p\leqslant n} (d_y - m_{i_1} - m_{i_2} - \ldots - m_{i_p})\Big)\tau = 0,$$

where $\prod$ is a product, $m_{i_j}^2 = 3 - k_{i_j}^2$, and k_{i_j} are any constants. The relationships given above can also be used to construct solutions.

We turn to the construction of vortex structures. The functions u_i corresponding to the soliton solutions τ_i found are calculated in accordance with (5). Level lines of the functions u_i for $i = 2$, 3, and 4 are given in Figs. 1, 2, and 3, respectively. The two heavier lines correspond to values of 0 and -1 for u_i. The isolines in Fig. 1 were obtained for constants $s_1 = s_2 = 1$, $k_1 = 1.25$, and $k_2 = 1$; and those in Fig. 2 for $s_1 = s_2 = s_3 = 1$, $k_1 = 1.25$, $k_2 = 1$, and $k_3 = 0.5$; those in Fig. 3 for $s_i = 1$ $(i = 1, \ldots, 4)$, $k_1 = 1$, $k_2 = 0.8$, $k_3 = 1.4$, and $k_4 = 1.6$.

For each function u_i we consider two sets in the plane of the flow:

$$L_n = \{(x,y)\colon\ u_i(x,y) < 0\}, \qquad L_p = \{(x,y)\colon\ u_i(x,y) > 0\}.$$

The level lines lying in L_n or L_p coincide with streamlines, so Figs. 1–3 represent flow patterns in the $R^2(x,y)$ plane. If the set

$$S_{ab} = \{(x,y)\colon\ -1 < a \leqslant u_i(x,y) \leqslant b < 0\}$$

is compact (which is achieved by the choice of the numbers a and b), then, taking its boundary as solid walls, we can state that in each case, the flow in S_{ab} is stable against two-dimensional perturbations. In fact, the following relationships are valid for S_{ab} for certain positive numbers A and B:

$$A \leqslant \omega'(\psi) = \delta \exp(\psi) + 2\exp(-2\psi) \leqslant B.$$

The conditions of Arnol'd's theorem [8] on the stability of solutions against two-dimensional perturbations are thus satisfied. The vortex structure corresponding to the solution (12) is given in Fig. 4 ($r_1 = r_2 = 1$).

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